# Finite Amplitude Free Convection as an Initial Value Problem—I

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ABSTRACT

The Oberbeck-Boussinesq equations are reduced to a two-dimensional form governing "roll" convection between two free surfaces maintained at a constant temperature difference. These equations are then transformed to a set of ordinary differential equations governing the time variations of the double-Fourier coefficients for the motion and temperature fields. Non-linear transfer processes are retained and appear as quadratic interactions between the Fourier coefficients. Energy and heat transfer relations appropriate to this Fourier resolution, and a numerical method for solution from arbitrary initial conditions are given. As examples of the method, numerical solutions for a highly truncated Fourier representation are presented. These solutions, which are for a fixed Prandtl number and variable Rayleigh numbers, show the transient growth of convection from small perturbations, and in all cases studied approach steady states. The steady states obtained agree favorably with steady-state solutions obtained by previous investigators.

## 1. Introduction

With but little exception, the motions of the atmosphere, on all scales, are of convective origin. This is to say that the primary causes for air motions are the thermal inequalities which are constantly being imposed upon the atmosphere, mainly by solar heating. The particular forms which these motions take vary greatly in scale and character, ranging from chaotic thermal "turbulence" to highly organized systems such as hurricanes. For all cases, however, there is, among others, the common property that the motions which develop will transport heat and vorticity (or momentum) and it is these processes which introduce a basic non-linear content to atmospheric behavior.

As a first step towards understanding the complicated forms of this non-linearity it seems necessary to study model systems of much greater simplicity than are actually encountered. One class of such simple systems, capable of elucidating the non-linear properties of the convective process, is that formed by representing the spatial variations of the motion and temperature which evolve in Bénard-type experiments by a fixed and limited number of Fourier components. Similar Fourier methods have already been applied to the steady-state aspects of the Bénard-convection problem by Malkus and Veronis (1958), Kuo and Platzman (1960), and Kuo (1960), and have been applied extensively to other hydrodynamical and meteorological problems (e.g., Kampé de Fériet, 1948; Gambo *et al.*, 1955; Wippermann, 1956; Lorenz, 1960, 1962; Baer, 1961; Baer and Platzman, 1961; Bryan, 1959; and Saltzman, 1959).

We propose now to extend the application of this method to the case of time-dependent convective motions, using the same two-dimensional geometrical framework as considered by Malkus and Veronis (1958), Kuo and Platzman (1961), Malkus and Witt (1960), and Kuo (1961). Our aim in this first article is primarily to set forth the procedure, i.e., to formulate the mathematical model and method of solution. Solutions for a series of very simple cases in which the number of degrees of freedom is greatly restricted, will be given as examples. These solutions, which are for variable Rayleigh numbers, show the evolution of convection from small perturbations to a finite-amplitude steady state, and include as a special case the marginally unstable condition studied by Rayleigh (1916).

## 2. The governing equations

Let us define symbols as follows:

$x,y$ = horizontal coordinates
$z$ = vertical coordinate
$t$ = time
$u = dx/dt$
$v = dy/dt$
$w = dz/dt$
$\rho$ = density
$p$ = pressure
$T$ = temperature
$\nu$ = kinematic viscosity
$\kappa$ = coefficient of thermal diffusivity
$g$ = acceleration of gravity
$\epsilon$ = coefficient of volume expansion
$H$ = height of fluid
$\bar{f}$ = average of $f$ over a horizontal plane
$f' = f - \bar{f}$
$(f)_{\text{av}}$ = average of $f$ over the entire fluid

$f_1 = f - (f)_{av}$
$f_0$ = initial value of $f$
$\bar{p}_{h0} = g\rho_{av}(z-H)$ = hydrostatic pressure corresponding to $(\rho)_{av}$
$P = (p - \bar{p}_{h0})(\rho)_{av}^{-1}$

Then, according to the approximations of Oberbeck (1879) and Boussinesq (1903) we can write the equations governing convection in a liquid in the form,

$$\frac{du}{dt} + \frac{\partial P}{\partial x} - \nu\nabla^2 u = 0, \qquad (1)$$

$$\frac{dv}{dt} + \frac{\partial P}{\partial y} - \nu\nabla^2 v = 0, \qquad (2)$$

$$\frac{dw}{dt} + \frac{\partial P}{\partial z} - g\epsilon T_1 - \nu\nabla^2 w = 0, \qquad (3)$$

$$\frac{dT_1}{dt} - \kappa\nabla^2 T_1 = 0, \qquad (4)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \qquad (5)$$

In this system we have made use of the relation,

$$\rho^{-1} = (\rho)_{av}^{-1}(1+\epsilon T_1), \qquad (6)$$

which is the equation of state for our problem.

The liquid under consideration is taken to be of height $H$, with a rigid lower boundary and a free or rigid upper boundary, between which a temperature contrast $\Delta T_0 = \bar{T}_0(0) - \bar{T}_0(H)$ is maintained externally. To simplify the problem we shall constrain the convective motions to develop only in the form of two-dimensional "rolls" in the $x$-$z$ plane (i.e., $v = \partial/\partial y = 0$).

In this case the governing equations become,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{\partial P}{\partial x} - \nu\nabla^2 u = 0, \qquad (7)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + \frac{\partial P}{\partial z} - g\epsilon T_1 - \nu\nabla^2 w = 0, \qquad (8)$$

$$\frac{\partial T_1}{\partial t} + u\frac{\partial T_1}{\partial x} + w\frac{\partial T_1}{\partial z} - \kappa\nabla^2 T_1 = 0, \qquad (9)$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \qquad (10)$$

By virtue of (10) we can define a stream function $\psi$ as follows:

$$u = -\frac{\partial\psi}{\partial z}, \quad w = \frac{\partial\psi}{\partial x}. \qquad (11)$$

The temperature departure $T_1$ can be expanded into an average value along the horizontal and a departure therefrom, according to the relation,

$$T_1(x,z,t) = \bar{T}_1(z,t) + T_1'(x,z,t), \qquad (12)$$

and, in turn, we can expand $\bar{T}_1$ into a part representing a linear variation between the upper and lower boundary and a departure from this linear variation which we call $\bar{T}_1''$, i.e.,

$$\bar{T}_1(z,t) = \left[\bar{T}_1(0,t) - \frac{\Delta T_0}{H}z\right] + \bar{T}_1''(z,t). \qquad (13)$$

If we substitute (13) in (12) we obtain

$$T_1(x,z,t) = \left[\bar{T}_1(0,t) - \frac{\Delta T_0}{H}z\right] + \theta, \qquad (14)$$

where

$$\theta = \bar{T}_1''(z,t) + T_1'(x,z,t). \qquad (15)$$

For this model we shall assume that the temperature at the upper and lower boundaries are kept constant by external heating (i.e., $\partial\bar{T}_1(0)\partial t = \partial\bar{T}_1(H)/\partial t = 0$). Hence, if we eliminate $P$ from (7) and (8) by forming the vorticity equation for our problem, and introduce (11) and (14) we obtain,

$$\frac{\partial}{\partial t}\nabla^2\psi - \frac{\partial\psi}{\partial z}\frac{\partial}{\partial x}\nabla^2\psi + \frac{\partial\psi}{\partial x}\frac{\partial}{\partial z}\nabla^2\psi - g\epsilon\frac{\partial\theta}{\partial x} - \nu\nabla^4\psi = 0, \qquad (16)$$

$$\frac{\partial\theta}{\partial t} - \frac{\partial\psi}{\partial z}\frac{\partial\theta}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial z} - \frac{\Delta T_0}{H}\frac{\partial\psi}{\partial x} - \kappa\nabla^2\theta = 0. \qquad (17)$$

These are the governing equations for our model (cf., Malkus and Witt, 1960). We note that $\nabla^2\psi$ represents the vorticity of the motions in the $x$-$z$ plane (i.e., $\nabla^2\psi = \partial u/\partial z - \partial w/\partial x$), and that $\nabla^4\psi = \nabla^2\nabla^2\psi = \partial^4\psi/\partial x^4 + \partial^4\psi/\partial z^4 + 2\partial^4\psi/\partial x^2\partial z^2$. We can introduce a further notational simplification by writing the non-linear advective terms in the form of a Jacobian operator,

$$\frac{\partial(a,b)}{\partial(x,z)} = \left(\frac{\partial a}{\partial x}\frac{\partial b}{\partial z} - \frac{\partial b}{\partial x}\frac{\partial a}{\partial z}\right)$$

in which case (16) and (17) take the form

$$\frac{\partial}{\partial t}\nabla^2\psi + \frac{\partial(\psi,\nabla^2\psi)}{\partial(x,z)} - g\epsilon\frac{\partial\theta}{\partial x} - \nu\nabla^4\psi = 0, \qquad (16')$$

$$\frac{\partial\theta}{\partial t} + \frac{\partial(\psi,\theta)}{\partial(x,z)} - \frac{\Delta T_0}{H}\frac{\partial\psi}{\partial x} - \kappa\nabla^2\theta = 0. \qquad (17')$$

Following the procedure of Malkus and Veronis (1958) for example, we shall measure length in units of $H$, time in units of $(H^2/\kappa)$, and temperature in units of $(\kappa\nu/g\epsilon H^3)$ so that we can rewrite the variables of the problem in terms of non-dimensional variables, to be denoted by an asterisk, as follows:

$$\begin{aligned} x &= Hx^* \\ z &= Hz^* \\ t &= (H^2/\kappa)t^* \\ \nabla^2 &= (1/H^2)\nabla^{*2} \\ \psi &= \kappa\psi^* \\ \theta &= (\kappa\nu/g\epsilon H^3)\theta^*. \end{aligned} \tag{18}$$

By the introduction of these transformations into (16′) and (17′) we obtain the non-dimensional equations,

$$\nabla^{*2}\frac{\partial\psi^*}{\partial t^*}+\frac{\partial(\psi^*,\nabla^{*2}\psi^*)}{\partial(x^*,z^*)}-\sigma\frac{\partial\theta^*}{\partial x^*}-\sigma\nabla^{*4}\psi^*=0, \tag{19}$$

$$\frac{\partial\theta^*}{\partial t^*}+\frac{\partial(\psi^*,\theta^*)}{\partial(x^*,z^*)}-R\frac{\partial\psi^*}{\partial x^*}-\nabla^{*2}\theta^*=0, \tag{20}$$

where

$\sigma=\dfrac{\nu}{\kappa}$ (The Prandtl Number)

$R=\dfrac{g\epsilon H^3\Delta T_0}{\kappa\nu}$ (The Rayleigh Number).

*The boundary conditions.* At a "free" (no-stress) boundary the vertical velocity and tangential stress vanish so that

$$\psi=\text{a constant}=0$$

and

$$\frac{\partial u}{\partial z}=0,$$

which implies that

$$\nabla^2\psi=0, \tag{21}$$

or

$$\psi^*=\nabla^{*2}\psi^*=0.$$

At a rigid (no-slip) boundary the vertical velocity and the tangential velocity vanish so that

$$\psi=0$$

and

$$\frac{\partial\psi}{\partial z}=0, \tag{22}$$

or

$$\psi^*=\frac{\partial\psi^*}{\partial z^*}=0.$$

At both the upper and lower boundaries, whether taken to be free or rigid, the temperature is assumed to be maintained at a constant value, so that

$$\theta=\theta^*=0. \tag{23}$$

It is implied that at the upper and lower boundaries

$$\frac{\partial(\psi^*,\nabla^{*2}\psi^*)}{\partial(x^*,z^*)}=\frac{\partial(\psi^*,\theta^*)}{\partial(x^*,z^*)}=0.$$

We shall here adapt the free boundary condition at both upper and lower boundaries (cf., Kuo and Platzman, 1961) and we shall dispense with lateral boundaries by considering that the liquid extends to infinity in the horizontal.

### 3. The Fourier representation

Let us assume that the stream function and temperature departure can be represented as a sum of double-Fourier components having a fundamental wave-length $L$ in the $x$-direction and $2H$ in the $z$-direction. Formally, we can then expand $\psi^*$ and $\theta^*$ as follows:

$$\psi^*(x^*,z^*,t^*)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\Psi(m,n,t^*)\exp\left[2\pi Hi\left(\frac{m}{L}x^*+\frac{n}{2H}z^*\right)\right], \tag{24}$$

$$\theta^*(x^*,z^*,t^*)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\Theta(m,n,t^*)\exp\left[2\pi Hi\left(\frac{m}{L}x^*+\frac{n}{2H}z^*\right)\right], \tag{25}$$

where $m$ is the wave number in the $x$-direction, $n$ is the wave number in the $z$-direction, and the complex Fourier coefficients are given by

$$\Psi(m,n,t^*)=\frac{1}{2LH}\int_0^L\int_{-H}^{H}\psi^*(x^*,z^*,t^*)\exp\left[-2\pi Hi\left(\frac{m}{L}x^*+\frac{n}{2H}z^*\right)\right]dxdz, \tag{26}$$

$$\Theta(m,n,t^*)=\frac{1}{2LH}\int_0^L\int_{-H}^{H}\theta^*(x^*,z^*,t^*)\exp\left[-2\pi Hi\left(\frac{m}{L}x^*+\frac{n}{2H}z^*\right)\right]dxdz. \tag{27}$$

Thus, by requiring, as the free boundary condition, that $z=0$ be a node for $\psi^*$ and $\theta^*$, we can represent cellular convection in the region $z=0$ to $H$.

In order to obtain equations governing the Fourier coefficients we transform (19) and (20) by multiplying these equations by

$$\frac{1}{2LH}\exp\left[-2\pi Hi\left(\frac{m}{L}x^*+\frac{n}{2H}z^*\right)\right]$$

and integrating over the fundamental region $2LH$. Then if we apply the Fourier transform relations (24) to (27) we obtain the following set of ordinary differential equations:

$$\dot{\Psi}(m,n)=\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\frac{C(m,n,p,q)\alpha^2(p,q)}{\alpha^2(m,n)}\Psi(p,q)\Psi(m-p,\,n-q)-\frac{\sigma i l^* m}{\alpha^2(m,n)}\Theta(m,n)-\sigma\alpha^2(m,n)\Psi(m,n) \tag{28}$$

$$\dot{\Theta}(m,n)=-\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}C(m,n,p,q)\Psi(p,q)\Theta(m-p,\,n-q)+Rl^*mi\Psi(m,n)-\alpha^2(m,n)\Theta(m,n) \tag{29}$$

where $C(m,n,p,q)=l^*h^*(mq-np)$, $l^*=2\pi H/L$, $h^*=\pi$, $\alpha^2(a,b)=(l^{*2}a^2+h^{*2}b^2)=(2\pi H/L)^2a^2+\pi^2b^2$, and $(\dot{\ })=d(\ )/dt^*$.

If we write $\Psi$ and $\Theta$ in terms of their real and imaginary parts, respectively, as follows,

$$\Psi(m,n)=\Psi_1(m,n)-i\Psi_2(m,n) \tag{30}$$

$$\Theta(m,n)=\Theta_1(m,n)-i\Theta_2(m,n), \tag{31}$$

we can write (28) and (29) as follows:

$$\dot{\Psi}_1(m,n)=\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}C(m,n,p,q)\frac{\alpha^2(p,q)}{\alpha^2(m,n)}[\Psi_1(p,q)\Psi_1(m-p,\,n-q)-\Psi_2(p,q)\Psi_2(m-p,\,n-q)]-\frac{\sigma l^* m}{\alpha^2(m,n)}\Theta_2(m,n)-\sigma\alpha^2(m,n)\Psi_1(m,n) \tag{32}$$

$$\dot{\Psi}_2(m,n)=\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}C(m,n,p,q)\frac{\alpha^2(p,q)}{\alpha^2(m,n)}[\Psi_1(p,q)\Psi_2(m-p,\,n-q)+\Psi_2(p,q)\Psi_1(m-p,\,n-q)]+\frac{\sigma l^* m}{\alpha^2(m,n)}\Theta_1(m,n)-\sigma\alpha^2(m,n)\Psi_2(m,n) \tag{33}$$

$$\dot{\Theta}_1(m,n)=-\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}C(m,n,p,q)[\Psi_1(p,q)\Theta_1(m-p,\,n-q)-\Psi_2(p,q)\Theta_2(m-p,\,n-q)]+Rl^*m\Psi_2(m,n)-\alpha^2(m,n)\Theta_1(m,n) \tag{34}$$

$$\dot{\Theta}_2(m,n)=-\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}C(m,n,p,q)[\Psi_1(p,q)\Theta_2(m-p,\,n-q)+\Psi_2(p,q)\Theta_1(m-p,\,n-q)]-Rl^*m\Psi_1(m,n)-\alpha^2(m,n)\Theta_2(m,n). \tag{35}$$

From the definitions (26), (27), (30) and (31) we have,

$$\Psi_1(m,n)=\Psi_1(-m,\,-n) \tag{36}$$

$$\Psi_2(m,n)=-\Psi_2(-m,\,-n) \tag{37}$$

$$\Theta_1(m,n)=\Theta_1(-m,\,-n) \tag{38}$$

$$\Theta_2(m,n)=-\Theta_2(-m,\,-n), \tag{39}$$

and from the free boundary conditions (21) and (23) applying at $z=0$, $H$, we have,

$$\Psi_1(m,n)=-\Psi_1(m,\,-n)\equiv-\Psi_1(-m,\,n) \tag{40}$$

$$\Psi_2(m,n)=-\Psi_2(m,\,-n)\equiv\Psi_2(-m,\,n) \tag{41}$$

$$\Theta_1(m,n)=-\Theta_1(m,\,-n)\equiv-\Theta_1(-m,\,n) \tag{42}$$

$$\Theta_2(m,n)=-\Theta_2(m,\,-n)\equiv\Theta_2(-m,\,n). \tag{43}$$

As special cases of (36) to (43) we have

$$\Psi_1(0,n)=\Theta_1(0,n)=\Psi_1(m,0)=\Psi_2(m,0)=\Theta_1(m,0)=\Theta_2(m,0)=0.$$

Eq (32) to (43) are the basic Fourier relations which constitute the simple convection model to be studied here. Note that in the stream function Fourier Equations (32) and (33), the quadratic terms represent nonlinear interactions among the spectral components of the motion field, the first of the two linear terms represents the effects of bouyancy in generating motion, and the second linear term represents viscous damping. In the temperature Fourier Equations (34) and (35), the quadratic terms represent non-linear heat transports associated with interaction among spectral components of the motion and temperature fields, the first of the two linear terms represents the effects of the basic heat transfer, and the second linear term represents the damping effect of conduction.

We shall next set down the general energy equations and heat transfer relations appropriate to the model.

## 4. Energy and heat transfer equations

We define two forms of energy averaged over the fundamental region, measured per unit mass:

$$K=\left[\frac{u^2+w^2}{2}\right]_{av}=\frac{1}{2}\left[\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial z}\right)^2\right]_{av} \quad \text{(kinetic energy)}, \qquad (44)$$

$$A=-\frac{g\epsilon H}{2\Delta T_0}[\theta^2]_{av} \quad \text{(available potential energy)}. \qquad (45)$$

Note that the available energy, $A$, is a maximum when $[\theta^2]_{av}=0$, which means that any departures from the initial linear variation of temperature between $z=0$ and $H$ constitute a diminution of $A$ below the undisturbed condition, where $A=0$.

We may further resolve both $K$ and $A$ into components representing, respectively, the energies of the mean vertical stratification ($K_V$ and $A_V$) and the energies of the mean horizontal variations along $x$ ($K_H$ and $A_H$). Thus,

$$K=K_V+K_H \qquad (46)$$

$$A=A_V+A_H, \qquad (47)$$

where

$$K_V=\left[\frac{\bar{u}^2+\bar{w}^2}{2}\right]_{av}=\frac{\kappa^2}{H^2}\left[\frac{\bar{u}^{*2}+\bar{w}^{*2}}{2}\right]_{av}$$

$$K_H=\left[\frac{u'^2+w'^2}{2}\right]_{av}=\frac{\kappa^2}{H^2}\left[\frac{u^{*\prime 2}+w^{*\prime 2}}{2}\right]_{av}$$

$$A_V=\left[-\frac{g\epsilon H}{2\Delta T_0}\bar{\theta}^2\right]_{av}=-\frac{\kappa^2\sigma}{H^2R}\left[\frac{\bar{\theta}^{*2}}{2}\right]_{av}$$

$$A_H=\left[-\frac{g\epsilon H}{2\Delta T_0}\theta'^2\right]_{av}=-\frac{\kappa^2\sigma}{H^2R}\left[\frac{\theta^{*\prime 2}}{2}\right]_{av}$$

($u^*=-\partial\psi^*/\partial z^*$ and $w^*=\partial\psi^*/\partial x^*$). Since we shall exclude laminar flows along $x$ from present considerations (i.e., $\bar{u}=\bar{w}=K_V=0$), we can take $K=K_H$. From (7), (8), (9), (10) and (14) we can then write energy equations in the form:

$$\frac{dK}{dt}=\{A_H\cdot K\}-D \qquad (48)$$

$$\frac{dA_V}{dt}=-\{A_H\cdot A_V\}+G_V \qquad (49)$$

$$\frac{dA_H}{dt}=\{A_H\cdot A_V\}-\{A_H\cdot K\}+G_H \qquad (50)$$

where

$$\{A_H\cdot K\}=g\epsilon(w\theta)_{av}=\frac{\kappa^3}{H^4}\sigma(w^*\theta^*)_{av}$$

$$\{A_H\cdot A_V\}=\frac{g\epsilon H}{\Delta T_0}\left(w\theta\frac{\partial\bar{\theta}}{\partial z}\right)_{av}=\frac{\kappa^3}{H^4}\frac{\sigma}{R}\left[w^*\theta^*\frac{\partial\bar{\theta}^*}{\partial z^*}\right]_{av}$$

$$D=-\nu(u\nabla^2u+w\nabla^2w)_{av}$$
$$=-\frac{\kappa^3}{H^4}\sigma(u^*\nabla^{*2}u^*+w^*\nabla^{*2}w^*)_{av}$$

$$G_V=-\frac{\kappa g\epsilon H}{\Delta T_0}(\bar{\theta}\nabla^2\bar{\theta})_{av}=-\frac{\kappa^3}{H^4}\frac{\sigma}{R}(\bar{\theta}^*\nabla^{*2}\bar{\theta}^*)_{av}$$

$$G_H=-\frac{\kappa g\epsilon H}{\Delta T_0}(\theta\nabla^2\theta-\bar{\theta}\nabla^2\bar{\theta})_{av}=-\frac{\kappa g\epsilon H}{\Delta T_0}(\theta'\nabla^2\theta')_{av}$$
$$=-\frac{\kappa^3}{H^4}\frac{\sigma}{R}(\theta^{*\prime}\nabla^{*2}\theta^{*\prime})_{av}.$$

In terms of the *non-dimensional* variables we can define

$$K^*=\frac{H^2}{\kappa^2}K=\left[\frac{u^{*2}+w^{*2}}{2}\right]_{av} \qquad (51)$$

$$A_V^*=\frac{H^2}{\kappa^2}A_V=-\frac{\sigma}{2R}(\bar{\theta}^{*2})_{av} \qquad (52)$$

$$A_H^*=\frac{H^2}{\kappa^2}A_H=-\frac{\sigma}{2R}(\theta'^{*2})_{av}, \qquad (53)$$

which, by applying (24)–(27), can be expanded into spectral components as follows:

$$K^*=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\tfrac{1}{2}\mathcal{K}(m,n) \qquad (54)$$

where

$$\mathcal{K}(m,n)=\alpha^2(m,n)|\Psi(m,n)|^2,$$

or expanding further, we have:

$$K^*=K_V^*+K_H^* \tag{55}$$

$$K_V^*=\sum_{n=1}^{\infty}\mathcal{K}(0,n) \tag{56}$$

$$K_H^*=\sum_{m=1}^{\infty}\{\mathcal{K}(m,0)+\sum_{n=1}^{\infty}[\mathcal{K}(m,n)+\mathcal{K}(m,-n)]. \tag{57}$$

$$A^*=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\tfrac{1}{2}\mathcal{A}(m,n) \tag{58}$$

where

$$\mathcal{A}(m,n)=-\frac{\sigma}{R}|\Theta(m,n)|^2,$$

or expanding further, we have,

$$A^*=A_V^*+A_H^* \tag{59}$$

$$A_V^*=\sum_{n=1}^{\infty}\mathcal{A}(0,n) \tag{60}$$

$$A_H^*=\sum_{m=1}^{\infty}\{\mathcal{A}(m,0)+\sum_{n=1}^{\infty}[\mathcal{A}(m,n)+\mathcal{A}(m,-n)]. \tag{61}$$

If now we apply the boundary conditions (40)–(43), we have the relations $\mathcal{K}(m,n)=\mathcal{K}(m,-n)$ and $\mathcal{A}(m,n)=\mathcal{A}(m,-n)$ so that,

$$K_H^*=\sum_{m=1}^{\infty}\{\mathcal{K}(m,0)+\sum_{n=1}^{\infty}2\mathcal{K}(m,n)\} \tag{62}$$

$$A_H^*=\sum_{m=1}^{\infty}\{\mathcal{A}(m,0)+\sum_{n=1}^{\infty}2\mathcal{A}(m,n)\}. \tag{63}$$

From (28) and (29) we obtain the following equations for the rates of the change of $\mathcal{K}(m,n)$ and $\mathcal{A}(m,n)$:

$$\frac{d\mathcal{K}(m,n)}{dt}=\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}C(m,n,p,q)\alpha^2(p,q)[\Psi(-m,-n)\Psi(p,q)\Psi(m-p,n-q)-\Psi(m,n)\Psi(p,q)\Psi(-m-p,-n-q)]$$
$$-\sigma il^*m[\Psi(-m,-n)\Theta(m,n)-\Psi(m,n)\Theta(-m,-n)]-2\sigma\alpha^2(m,n)\mathcal{K}(m,n) \tag{64}$$

$$\frac{d\mathcal{A}(m,n)}{dt}=\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\frac{\sigma}{R}C(m,n,p,q)[\Theta(-m,-n)\Psi(p,q)\Theta(m-p,n-q)-\Theta(m,n)\Psi(p,q)\Theta(-m-p,-n-q)]$$
$$+\sigma il^*m[\Psi(-m,-n)\Theta(m,n)-\Psi(m,n)\Theta(-m,-n)]-2\alpha^2(m,n)\mathcal{A}(m,n). \tag{65}$$

We now set down several relationships for the vertical heat transfer. From (9) and (10) we have

$$\frac{\partial\bar{T}_1}{\partial t}+\frac{\partial J}{\partial z}=0, \tag{66}$$

where

$$J=\overline{wT_1}-\kappa\frac{\partial\bar{T}_1}{\partial z}, \tag{67}$$

which is the total rate of vertical heat transfer by both convection (first term) and conduction (second term).

If we let the subscript $s$ denote the *steady-state* condition, we have

$$\frac{\partial J_s}{\partial z}=0, \tag{68}$$

from which it follows, by integration from the lower boundary $z=0$ to an arbitrary level $z=Z$ and application of the lower boundary condition, that

$$J_s(Z)=J_s(0)=-\kappa\left[\left(\frac{\partial\bar{T}_1}{\partial z}\right)_{z=0}\right]_s=\kappa\left[\frac{\Delta T_0}{H}-\left(\frac{\partial\bar{\theta}}{\partial z}\right)_{z=0}\right]_s. \tag{69}$$

In terms of the non-dimensional variables (18),

$$J_s=\frac{\kappa^2\nu}{g\epsilon H^4}\left[R-\left(\frac{\partial\bar{\theta}^*}{\partial z^*}\right)_{z=0}\right]_s. \tag{70}$$

The first term, which we denote by

$$J_s'=\frac{\kappa^2\nu}{g\epsilon H^4}R=\frac{\kappa\Delta T_0}{H},$$

is the steady-state heat transfer in the absence of convection, and the second term,

$$J_s''=-\kappa\left(\frac{\partial\bar{\theta}}{\partial z}\right)_{z=0}=\frac{\kappa^2\nu}{g\epsilon H^4}\left(\frac{\partial\bar{\theta}^*}{\partial z^*}\right)_{z=0},$$

is the component due to the presence of convection. This latter component is a function of $R$ and is to be determined by solution of (19) and (20) subject to the boundary conditions.

It is usually of interest to measure the importance of the convective motions in transporting heat by the ratio called the Nusselt Number

$$N_s=\frac{J_s}{J_s'}$$

$$=1-\frac{1}{R}\left(\frac{\partial\bar{\theta}^*}{\partial z^*}\right)_{z=0}, \quad (71)$$

or the heat transfer ratio, $S$, given by $S=\lambda N_s$ (cf., Kuo, 1961), where $\lambda$ is defined below.

According to the Fourier expansion (25) we have

$$\bar{\theta}^*=\sum_{n=-\infty}^{\infty}\Theta(0,n)\exp[i\pi nz^*]$$

$$=\sum_{n=1}^{\infty}[2\Theta_1(0,n)\cos\pi nz^*+2\Theta_2(0,n)\sin\pi nz^*]. \quad (72)$$

The boundary conditions require that $\Theta_1(0,n)=0$ so that we can write,

$$\bar{\theta}^*=\sum_{n=1}^{\infty}2\Theta_2(0,n)\sin\pi nz^*, \quad (73)$$

from which it follows that

$$\frac{\partial\bar{\theta}^*}{\partial z^*}=\sum_{n=1}^{\infty}2\pi n\Theta_2(0,n)\cos\pi nz^*$$

and

$$J_s=\frac{\kappa^2\nu}{g\epsilon H^4}[R-2\pi\sum_{n=1}^{\infty}n\Theta_2(0,n)]_s \quad (74)$$

$$N_s=1-\frac{2\pi}{R}\sum_{n=1}^{\infty}n[\Theta(0,n)]_s. \quad (75)$$

For a pure cellular convection, $n=2$ and

$$J_s=\frac{\kappa^2\nu}{g\epsilon H^4}[R-4\pi\Theta_2(0,2)]_s, \quad (76)$$

in which case the steady-state rate of heat transport due to the presence of convection is given by

$$J_s''=-\frac{4\pi\kappa^2\nu}{g\epsilon H^4}[\Theta_2(0,2)]_s, \quad (77)$$

and

$$N_s=1-\frac{4\pi}{R}[\Theta_2(0,2)]_s. \quad (78)$$

The mean temperature profile in the vertical can be written in the form,

$$\bar{T}_1(z^*)=\bar{T}_1(0)+\Delta T_0Q(z^*), \quad (79)$$

where

$$Q(z^*)=\frac{\bar{T}_1^*(z^*)-\bar{T}_1^*(0)}{R}$$

$$=-\left[z^*-\frac{1}{R}\bar{\theta}^*\right]$$

$$=-\left[z^*-\frac{2}{R}\sum_{n=1}^{\infty}\Theta_2(0,n)\sin n\pi z^*\right]. \quad (80)$$

## 5. A special model

In order to specialize the model we must now select the fundamental region by fixing the ratio $L/H$. We shall be guided in this selection by Rayleigh's (1916) solution of the eigenvalue problem posed by the linearized form of (19) and (20) (cf., Malkus and Veronis, 1958). The result for two "free" boundaries is that for a critical minimum value of the Rayleigh number,

$$R=R_c=\frac{27}{4}\pi^4,$$

a steady solution of the form

$$\psi^*=A\sin\frac{\pi}{\sqrt{2}}x^*\sin\pi z^*$$

$$\theta^*=B\cos\frac{\pi}{\sqrt{2}}x^*\sin\pi z^*$$

$$A,B=\text{constants}$$

obtains, representing cellular convection of horizontal wavelength equal to $2\sqrt{2}$ times the depth of the fluid.

We shall choose our fundamental region such that this Rayleigh solution corresponds to $m=3$ and $n=1$ in our Fourier expansion, which is to say $L/H=6\sqrt{2}$. Thus values of $m$ less than three represent horizontal cells larger than the critical Rayleigh mode and values of $m$ greater than three represent smaller horizontal cells.

At this time we have computed coefficients for a truncated system consisting of the components included by wave numbers $m\leq 6$ and $n\leq 2$. In the special case of $\Theta_2(0,n)$ (which represents the departure of the vertical temperature stratification from the basic linear variation) the additional wave numbers $n=3$ and 4 are included.

To complete the specialization we have assumed we are dealing with a liquid of Prandtl number, $\sigma=10$, which is about twice that of water ($\sigma=4.8$). The Rayleigh number will be treated as a variable parameter.

With these specifications we can write (32) to (43) as a set of 52 ordinary differential equations of the form

$$\frac{dX_i}{dt^*}=\sum_{j,k} C_{ijk}X_jX_k, \tag{81}$$

where $X_i$, $X_j$ and $X_k$ denote the variables $\Psi_1(m,n)$, $\Psi_2(m,n)$, $\Theta_1(m,n)$, and $\Theta_2(m,n)$ according to the subscript assignments given in Table 1, and $C_{ijk}$ denote the coefficients. The linear terms are represented by $k=0$, $X_0=1$. In terms of this notation the 52 equations can be written most conveniently in the form of a table (Table 2). In this table the values entered for the coefficients of the first of the two linear terms in the thermal equations ($i=25$ to 48) are for $R=R_c$. Values for other values of $R$ can be obtained simply by multiplying these coefficients by $\lambda=R/R_c$.

## 6. Numerical methods

The set of equations represented by (81) and Table 2 can be solved by numerical procedures as a "marching" problem, given the initial conditions. The particular procedure used here is the "double approximation forward difference method" used by Bryan (1957) and Saltzman (1959), for example.

Specifically, let $\Delta t^*$ be an increment of $t^*$, let $n$ be the number of such increments, and let the value of $X_i$ at $t^*=n\Delta t^*$ be denoted by $(X_i)_n$. To procede from $n$ to $n+1$ we first compute two preliminary approximations for the first and second steps beyond $n$:

$$(X_i)_{n+1}{}'=(X_i)_n+\Delta t^*\sum_{jk} C_{ijk}(X_j)_n(X_k)_n, \tag{82}$$

$$(X_i)_{n+2}{}'=(X_i)_{n+1}{}'+\Delta t^*\sum_{jk} C_{ijk}(X_j)_{n+1}{}'(X_k)_{n+1}{}'. \tag{83}$$

The second of these is then combined with $(X_i)_n$ to give $(X_i)_{n+1}$:

$$\begin{aligned}(X_i)_{n+1}&=\tfrac{1}{2}[(X_i)_n+(X_i)_{n+2}{}']\\&=\tfrac{1}{2}[(X_i)_n+(X_i)_{n+1}{}'\\&\qquad+\Delta t^*\sum_{jk} C_{ijk}(X_j)_{n+1}{}'(X_k)_{n+1}{}'].\end{aligned} \tag{84}$$

## 7. Examples of solutions for a highly truncated system

We now present some examples of numerical solutions for the growth of cellular convective motions from small perturbations. The following conditions apply to all the cases:

(1) The vertical nodal surfaces of the convection cells are fixed by excluding $\Psi_2(m,n)$ and $\Theta_1(m,n)$ for all $m$, $n$.

(2) The initial conditions consist of small perturbations of the stream field only, given numerically by $\Psi_1(m,n,0)=0.0005$. [$\Theta_2(m,n,0)=0$].

(3) The non-dimensional finite time increment is $\Delta t^*=0.001$. This permits the integrations to proceed to steady states without any computational instability.

(4) The only components permitted are the seven variables given in Table 1 as 5, 7, 13, 30, 32, 38 and 50. For convenience we shall assign letters to these variables as follows:

| Variable | |
|---|---|
| $5\equiv\Psi_1(3,1)$ | $\equiv A$ |
| $7\equiv\Psi_1(4,1)$ | $\equiv B$ |
| $13\equiv\Psi_1(1,2)$ | $\equiv C$ |
| $30\equiv\Theta_2(3,1)$ | $\equiv D$ |
| $32\equiv\Theta_2(4,1)$ | $\equiv E$ |
| $38\equiv\Theta_2(1,2)$ | $\equiv F$ |
| $50\equiv\Theta_2(0,2)$ | $\equiv G.$ |

Then, from Table 2, we can write the governing equations for $\sigma=10$ and variable Rayleigh number in the form,

$$\dot{A}=23.521BC-1.500D-148.046A$$
$$\dot{B}=-22.030AC-1.589E-186.429B$$
$$\dot{C}=1.561AB-0.185F-400.276C$$
$$\dot{D}=-16.284CE-16.284BF-13.958AG-1460.631\lambda A-14.805D$$

TABLE 1. Subscripts $i$, of $X_i$, assigned to the Fourier coefficient variables, $\Psi_1(m,n)$, $\Psi_2(m,n)$, $\Theta_1(m,n)$, and $\Theta_2(m,n)$.

| $(m,n)$ | (1,1) | (2,1) | (3,1) | (4,1) | (5,1) | (6,1) | (1,2) | (2,2) | (3,2) | (4,2) | (5,2) | (6,2) | (0,1) | (0,2) | (0,3) | (0,4) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\Psi_1$ | 1 | 3 | 5 | 7 | 9 | 11 | 13 | 15 | 17 | 19 | 21 | 23 | — | — | — | — |
| $\Psi_2$ | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 | — | — | — | — |
| $\Theta_1$ | 25 | 27 | 29 | 31 | 33 | 35 | 37 | 39 | 41 | 43 | 45 | 47 | — | — | — | — |
| $\Theta_2$ | 26 | 28 | 30 | 32 | 34 | 36 | 38 | 40 | 42 | 44 | 46 | 48 | 49 | 50 | 51 | 52 |

TABLE 2. Coefficients for Fourier equations, $\frac{dX_i}{dt^*} = \sum_{jk} C_{ijk}X_jX_k$. Coefficients of linear terms denoted by $k=0$, $X_0=1$. See Table 1.

| i= | 1 | | 2 | | 3 | | 4 | | 5 | | 6 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | C | j,k | C | j,k | C | j,k | C | j,k | C | j,k | C | j,k |
| | 36.851 | 11,21 | 36.851 | 12,21 | 28.761 | 11,19 | 28.761 | 12,19 | 20.935 | 11,17 | 20.935 | 12,17 |
| | 36.851 | 12,22 | −36.851 | 11,22 | 28.761 | 12,20 | −28.761 | 11,20 | 20.935 | 12,18 | −20.935 | 11,18 |
| | 33.055 | 9,19 | 33.055 | 10,19 | 28.126 | 9,17 | 28.126 | 10,17 | 22.744 | 9,15 | 22.744 | 10,15 |
| | 33.055 | 10,20 | −33.055 | 9,20 | 28.126 | 10,18 | −28.126 | 9,18 | 22.744 | 10,16 | −22.744 | 9,16 |
| | 28.770 | 7,17 | 28.770 | 8,17 | 26.646 | 7,15 | 26.646 | 8,15 | 23.521 | 7,13 | 23.521 | 8,13 |
| | 28.770 | 8,18 | −28.770 | 7,18 | 26.646 | 8,16 | −26.646 | 7,16 | 23.521 | 8,14 | −23.521 | 7,14 |
| | 23.986 | 5,15 | 23.996 | 6,15 | 24.320 | 5,13 | 24.320 | 6,13 | −21.970 | 3,13 | −21.970 | 4,13 |
| | 23.986 | 6,16 | −23.996 | 5,16 | 24.320 | 6,14 | −24.320 | 5,14 | 21.970 | 4,14 | −21.970 | 3,14 |
| | 18.732 | 3,13 | 18.732 | 4,13 | −17.130 | 1,13 | −17.130 | 2,13 | −19.643 | 1,15 | −19.643 | 2,15 |
| | 18.732 | 4,14 | −18.732 | 3,14 | 17.130 | 2,14 | −17.130 | 1,14 | 19.643 | 2,16 | −19.643 | 1,16 |
| | −7.224 | 3,17 | 7.224 | 4,17 | 6.556 | 1,17 | −6.556 | 2,17 | 11.889 | 1,19 | −11.889 | 2,19 |
| | −7.224 | 4,18 | −7.224 | 3,18 | 6.556 | 2,18 | 6.556 | 1,18 | 11.889 | 2,20 | 11.889 | 1,20 |
| | −14.936 | 5,19 | 14.936 | 6,19 | −7.402 | 5,21 | 7.402 | 6,21 | 6.462 | 3,21 | −6.462 | 4,21 |
| | −14.936 | 6,20 | −14.936 | 5,20 | −7.402 | 6,22 | −7.402 | 5,22 | 6.462 | 4,22 | 6.462 | 3,22 |
| | −23.139 | 7,21 | 23.139 | 8,21 | −15.650 | 7,23 | 15.650 | 8,23 | −1.500 | 30,0 | 1.500 | 29,0 |
| | −23.139 | 8,22 | −23.139 | 7,22 | −15.650 | 8,24 | −15.650 | 7,24 | −148.046 | 5,0 | −148.046 | 6,0 |
| | −31.831 | 9,23 | 31.831 | 10,23 | −1.228 | 28,0 | 1.288 | 27,0 | | | | |
| | −31.831 | 10,24 | −31.831 | 9,24 | −120.627 | 3,0 | −120.627 | 4,0 | | | | |
| | −0.711 | 26,0 | 0.711 | 25,0 | | | | | | | | |
| | −104.185 | 1,0 | −104.185 | 2,0 | | | | | | | | |

| i= | 7 | | 8 | | 9 | | 10 | | 11 | | 12 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | C | j,k | C | j,k | C | j,k | C | j,k | C | j,k | C | j,k |
| | 15.051 | 11,15 | 15.051 | 12,15 | 11.306 | 11,13 | 11.306 | 12,13 | −14.216 | 9,13 | −14.216 | 10,13 |
| | 15.051 | 12,16 | −15.051 | 11,16 | 11.306 | 12,14 | −11.306 | 11,14 | 14.216 | 10,14 | −14.216 | 9,14 |
| | 18.473 | 9,13 | 18.473 | 10,13 | −18.987 | 7,13 | −18.987 | 8,13 | −18.092 | 7,15 | −18.092 | 8,15 |
| | 18.473 | 10,14 | −18.473 | 9,14 | 18.987 | 8,14 | −18.987 | 7,14 | 18.092 | 8,16 | −18.092 | 7,16 |
| | −22.030 | 5,13 | −22.030 | 6,13 | −21.205 | 5,15 | −21.205 | 6,15 | −20.936 | 5,17 | −20.936 | 6,17 |
| | 22.030 | 6,14 | −22.030 | 5,14 | 21.205 | 6,16 | −21.205 | 5,16 | 20.936 | 6,18 | −20.936 | 5,18 |
| | −22.167 | 3,15 | −22.167 | 4,15 | −22.342 | 3,17 | −22.342 | 4,17 | −22.745 | 3,19 | −22.745 | 4,19 |
| | 22.167 | 4,16 | −22.167 | 3,16 | 22.342 | 4,18 | −22.342 | 3,18 | 22.745 | 4,20 | −22.745 | 3,20 |
| | −21.209 | 1,17 | −21.209 | 2,17 | −22.395 | 1,19 | −22.395 | 2,19 | −23.520 | 1,21 | −23.520 | 2,21 |
| | 21.209 | 2,18 | −21.209 | 1,18 | 22.395 | 2,20 | −22.395 | 1,20 | 23.520 | 2,22 | −23.520 | 1,22 |
| | 16.009 | 1,21 | −16.009 | 2,21 | 19.258 | 1,23 | −19.258 | 2,23 | −1.500 | 36,0 | 1.500 | 35,0 |
| | 16.009 | 2,22 | 16.009 | 1,22 | 19.258 | 2,24 | 19.258 | 1,24 | −296.091 | 11,0 | −296.091 | 12,0 |
| | 11.768 | 3,21 | −11.768 | 4,21 | −1.570 | 34,0 | 1.570 | 33,0 | | | | |
| | 11.768 | 4,22 | 11.768 | 3,22 | −235.777 | 9,0 | −235.777 | 10,0 | | | | |
| | −1.589 | 32,0 | 1.589 | 31,0 | | | | | | | | |
| | −186.429 | 7,0 | −186.429 | 8,0 | | | | | | | | |

| i= | 13 | | 14 | | 15 | | 16 | | 17 | | 18 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | C | j,k | C | j,k | C | j,k | C | j,k | C | j,k | C | j,k |
| | 3.856 | 9,11 | 3.856 | 9,12 | 6.117 | 7,11 | 6.117 | 7,12 | 6.979 | 5,11 | 6.979 | 5,12 |
| | 3.856 | 10,12 | −3.856 | 10,11 | 6.117 | 8,12 | −6.117 | 8,11 | 6.979 | 6,12 | −6.979 | 6,11 |
| | 2.581 | 7,9 | 2.581 | 7,10 | 3.915 | 5,9 | 3.915 | 5,10 | 4.222 | 3,19 | 4.222 | 3,10 |
| | 2.581 | 8,10 | −2.581 | 8,9 | 3.915 | 6,10 | −3.915 | 6,9 | 4.222 | 4,10 | −4.222 | 4,9 |
| | 1.561 | 5,7 | 1.561 | 5,8 | 2.202 | 3,7 | 2.202 | 3,8 | 2.154 | 1,7 | 2.154 | 1,8 |
| | 1.561 | 6,8 | −1.561 | 6,7 | 2.202 | 4,8 | −2.202 | 4,7 | 2.154 | 2,8 | −2.154 | 2,7 |
| | 0.797 | 3,5 | 0.797 | 3,6 | 0.979 | 1,5 | 0.979 | 1,6 | −0.086 | 1,3 | −0.086 | 1,4 |
| | 0.797 | 4,6 | −0.797 | 4,5 | 0.979 | 2,6 | −0.979 | 2,5 | 0.086 | 2,4 | −0.086 | 2,3 |
| | 0.287 | 1,3 | 0.287 | 1,4 | −0.355 | 40,0 | 0.355 | 39,0 | −0.500 | 42,0 | 0.500 | 41,0 |
| | 0.287 | 2,4 | −0.287 | 2,3 | −416.718 | 15,0 | −416.718 | 16,0 | −444.137 | 17,0 | −444.137 | 18,0 |
| | −0.185 | 38,0 | 0.185 | 37,0 | | | | | | | | |
| | −400.276 | 13,0 | −400.276 | 14,0 | | | | | | | | |

| i= | 19 | | 20 | | 21 | | 22 | | 23 | | 24 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | C | j,k | C | j,k | C | j,k | C | j,k | C | j,k | C | j,k |
| | 6.767 | 3,11 | 6.767 | 3,12 | 5.876 | 1,11 | 5.876 | 1,12 | −2.068 | 1,9 | −2.068 | 1,10 |
| | 6.767 | 4,12 | −6.767 | 4,11 | 5.876 | 2,12 | −5.876 | 2,11 | 2.068 | 2,10 | −2.068 | 2,9 |
| | 3.806 | 1,9 | 3.806 | 1,10 | −1.079 | 1,7 | −1.079 | 1,8 | −0.517 | 3,7 | −0.517 | 3,8 |
| | 3.806 | 2,10 | −3.806 | 2,9 | 1.079 | 2,8 | −1.079 | 2,7 | 0.517 | 4,8 | −0.517 | 4,7 |
| | −0.422 | 1,5 | −0.422 | 1,6 | −0.120 | 3,5 | −0.120 | 3,6 | −0.750 | 48,0 | 0.750 | 47,0 |
| | 0.422 | 2,6 | −0.422 | 2,5 | 0.120 | 4,6 | −0.120 | 4,5 | −592.182 | 23,0 | −592.182 | 24,0 |
| | −0.614 | 44,0 | 0.614 | 43,0 | −0.696 | 46,0 | 0.696 | 45,0 | | | | |
| | −482.520 | 19,0 | −482.520 | 20,0 | −531.868 | 21,0 | −531.868 | 22,0 | | | | |

TABLE 2 (*continued*).

| i= 25 C | j,k | 26 C | j,k | 27 C | j,k | 28 C | j,k | 29 C | j,k | 30 C | j,k |
|---|---|---|---|---|---|---|---|---|---|---|---|
| −16.284 | 21,35 | 16.284 | 21,36 | −18.610 | 19,35 | −18.610 | 19,36 | −20.936 | 17,35 | −20.936 | 17,36 |
| −16.284 | 22,36 | 16.284 | 22,35 | −18.610 | 20,36 | 18.610 | 20,35 | −20.936 | 18,36 | 20.936 | 18,35 |
| −13.958 | 19,33 | −13.958 | 19,34 | −16.284 | 17,33 | −16.284 | 17,34 | −18.610 | 15,33 | −18.610 | 15,34 |
| −13.958 | 20.34 | −13.958 | 20,33 | −16.284 | 18,34 | 16.284 | 18,33 | −18.610 | 16,34 | 18.610 | 16,33 |
| −11.631 | 17,31 | −11.631 | 17,32 | −13.958 | 15,31 | −13.958 | 15,32 | −16.284 | 13,31 | −16.284 | 13,32 |
| −11.631 | 18,32 | 11.631 | 18,31 | −13.958 | 16,32 | 13.958 | 16,31 | −16.284 | 14,32 | 16.284 | 14,31 |
| −9.305 | 15,29 | −9.305 | 15,30 | −11.631 | 13,29 | −11.631 | 13,30 | 11.631 | 13,27 | −11.631 | 13,28 |
| −9.305 | 16,30 | 9.305 | 16,29 | −11.631 | 14,30 | 11.631 | 14,29 | −11.631 | 14,28 | 11.631 | 14,27 |
| −6.979 | 13,27 | −6.979 | 13,28 | 6.979 | 13,25 | 6.979 | 13,26 | 9.305 | 15,25 | −9.305 | 15,26 |
| −6.979 | 14,28 | 6.979 | 14,27 | −6.979 | 14,26 | 6.979 | 14,25 | −9.305 | 16,26 | 9.305 | 16,25 |
| 2.326 | 17,27 | −2.326 | 17,28 | −2.326 | 17,25 | 2.326 | 17,26 | −4.652 | 19,25 | 4.652 | 19,26 |
| 2.326 | 18,28 | 2.326 | 18,27 | −2.326 | 18,26 | −2.326 | 18,25 | −4.652 | 20,26 | −4.652 | 20,25 |
| 4.652 | 19,29 | −4.652 | 19,30 | 2.326 | 21,29 | −2.326 | 21,30 | −2.326 | 21,27 | 2.326 | 21,28 |
| 4.652 | 20,30 | 4.652 | 20,29 | 2.326 | 22,30 | 2.326 | 22,29 | −2.326 | 22,28 | −2.326 | 22,27 |
| 6.979 | 21,31 | −6.979 | 21,32 | 4.653 | 23,31 | −4.653 | 23,32 | 2.326 | 3,45 | 2.326 | 3,46 |
| 6.979 | 22.32 | 6.979 | 22,31 | 4.653 | 24,32 | 4.653 | 24,31 | 2.326 | 4,46 | −2.326 | 4,45 |
| 9.305 | 23,33 | −9.305 | 23,34 | −4.652 | 7,47 | −4.652 | 7,48 | 4.652 | 1,43 | 4.652 | 1,44 |
| 9.305 | 24,34 | 9.305 | 24,33 | −4.652 | 8,48 | 4.652 | 8,47 | 4.652 | 2,44 | −4.652 | 2,43 |
| −9.305 | 9,47 | −9.305 | 9,48 | −2.326 | 5,45 | −2.326 | 5,46 | −9.305 | 1,39 | −9.305 | 1,40 |
| −9.305 | 10,48 | 9.305 | 10,47 | −2.326 | 6,46 | 2.326 | 6,45 | 9.305 | 2,40 | −9.305 | 2,39 |
| −6.978 | 7,45 | −6.978 | 7.46 | 2.326 | 1,41 | 2.326 | 1,42 | −11.631 | 3,37 | −11.631 | 3,38 |
| −6.978 | 8,46 | 6.978 | 8,45 | 2.326 | 2,42 | −2.326 | 2,41 | 11.631 | 4,38 | −11.631 | 4,37 |
| −4.652 | 5,43 | −4.652 | 5,44 | −6.979 | 1.37 | −6.979 | 1,38 | 16.284 | 7,37 | −16.284 | 7,38 |
| −4.652 | 6,44 | 4.652 | 6,43 | 6.979 | 2,38 | −6.979 | 2,37 | 16.284 | 8,38 | 16.284 | 8,37 |
| −2.326 | 3,41 | −2.326 | 3,42 | 11.631 | 5,37 | −11.631 | 5,38 | 18.610 | 9,39 | −18.610 | 9,40 |
| −2.326 | 4,42 | 2.326 | 4,41 | 11.631 | 6,38 | 11.631 | 6,37 | 18.610 | 10,40 | 18.610 | 10,39 |
| 6.979 | 3,37 | −6.979 | 3,38 | 13.958 | 7,39 | −13.958 | 7,40 | 20.937 | 11,41 | −20.937 | 11,42 |
| 6.979 | 4,38 | 6.979 | 4,37 | 13.958 | 8,40 | 13.958 | 8,39 | 20.937 | 12,42 | 20.937 | 12,41 |
| 9.305 | 5,39 | −9.305 | 5,40 | 16.284 | 9,41 | −16.284 | 9,42 | −6.979 | 18,49 | 6.979 | 17,49 |
| 9.305 | 6,40 | 9.305 | 6,39 | 16.284 | 10,42 | 16.284 | 10,41 | 13.958 | 6,50 | −13.958 | 5,50 |
| 11.632 | 7,41 | −11.632 | 7,42 | 18.610 | 11,43 | −18.610 | 11,44 | 20.937 | 18,51 | −20.937 | 17,51 |
| 11.632 | 8,42 | 11.632 | 8,41 | 18.610 | 12,44 | 18.610 | 12,43 | 1460.642 | 6,0 | −1460.642 | 5,0 |
| 13.958 | 9,43 | −13.958 | 9,44 | −4.652 | 16,49 | 4.652 | 15,49 | −14.805 | 29,0 | −14.805 | 30,0 |
| 13.958 | 10,44 | 13.958 | 10,43 | 9.305 | 4,50 | −9.305 | 3,50 | | | | |
| 16.284 | 11,45 | −16.284 | 11,46 | 13.958 | 16,51 | −13.958 | 15,51 | | | | |
| 16.284 | 12,46 | 16.284 | 12,45 | 973.761 | 4,0 | −973.761 | 3,0 | | | | |
| −2.326 | 14,49 | 2.326 | 13,49 | −12.063 | 27,0 | −12.063 | 28,0 | | | | |
| 4.652 | 2,50 | −4.652 | 1,50 | | | | | | | | |
| 6.979 | 14,51 | −6.979 | 13,51 | | | | | | | | |
| 486.881 | 2,0 | −486.881 | 1,0 | | | | | | | | |
| −10.418 | 25,0 | −10.418 | 26,0 | | | | | | | | |

| i= 31 C | j,k | 32 C | j,k | 33 C | j,k | 34 C | j,k | 35 C | j,k | 36 C | j,k |
|---|---|---|---|---|---|---|---|---|---|---|---|
| −23.263 | 15,35 | −23.263 | 15,36 | −25.589 | 13,35 | −25.589 | 13,36 | −25.589 | 13,33 | 25.589 | 13,34 |
| −23.263 | 16,36 | 23.263 | 16,35 | −25.589 | 14,36 | 25.589 | 14,35 | −25.589 | 14,34 | 25.589 | 14,33 |
| −20.936 | 13,33 | −20.936 | 13,34 | 20.936 | 13,31 | 20.936 | 13,32 | 23.263 | 15,31 | 23.263 | 15,32 |
| −20.936 | 14,34 | 29.936 | 14,33 | −20.936 | 14,32 | 20.936 | 14,31 | −23.263 | 16,32 | 23.363 | 16,31 |
| 16.284 | 13,29 | 16.284 | 13,30 | 18.610 | 15,29 | 18.610 | 15,30 | 20.936 | 17,29 | 20.936 | 17,30 |
| −16.284 | 14,30 | 16,284 | 14,29 | −18.610 | 16,30 | 18.610 | 16,29 | −20.936 | 18,30 | 20.936 | 18,29 |
| 13.958 | 15.27 | 13.958 | 15,28 | 16.284 | 17,27 | 16.284 | 17,28 | 18.610 | 19,27 | 18.610 | 19,28 |
| −13.958 | 16,28 | 13.958 | 16,27 | −16.284 | 18,28 | 16.284 | 18,27 | −18.610 | 20,28 | 18.610 | 20,27 |
| 11.631 | 17,25 | 11.631 | 17,26 | 13.958 | 19,25 | 13.958 | 19,26 | 16.284 | 21,25 | 16.284 | 21,26 |
| −11.631 | 18,26 | 11.631 | 18,25 | −13.958 | 20,26 | 13.958 | 20,25 | −16.284 | 22,26 | 16.284 | 22,25 |
| −6.968 | 21,25 | 6,968 | 21,26 | −9. 305 | 23,25 | 9.305 | 23,26 | −16.284 | 1,45 | −16.284 | 1,46 |
| −6.968 | 22,26 | −6.968 | 22,25 | −9 .305 | 24,26 | −9.305 | 24,25 | 16.284 | 2,46 | −16,284 | 2,45 |
| −4.653 | 23,27 | 4.653 | 23,28 | 9.305 | 1,47 | 9,305 | 1,48 | −18.610 | 3,43 | −18.610 | 3,44 |
| −4.653 | 24,28 | −4.653 | 24,27 | 9.305 | 2,48 | −9.305 | 2,47 | 18.610 | 4,44 | −18.610 | 4,43 |
| 4.652 | 3,47 | 4.652 | 3,48 | −13.958 | 1,43 | −13.958 | 1,44 | −20.936 | 5,41 | −20.936 | 5,42 |
| 4.652 | 4,48 | −4.652 | 4,47 | 13.958 | 2,44 | −13.958 | 2,43 | 20.936 | 6,42 | −20.936 | 6,41 |
| 6.979 | 1,45 | 6.979 | 1,46 | −16.283 | 3,41 | −16.283 | 3,42 | −23.262 | 7,39 | −23.262 | 7 40 |
| 6.979 | 2,46 | −6.979 | 2,45 | 16.283 | 4,42 | −16.283 | 4,41 | 23.262 | 8,40 | −23.262 | 8'39 |
| −11.632 | 1,41 | −11.632 | 1,42 | −18.610 | 5,39 | −18.610 | 5,40 | −25.589 | 9,37 | −25,589 | 9·38 |
| 11.632 | 2,42 | −11.632 | 2,41 | 18.610 | 6,40 | −18.610 | 6,39 | 25.589 | 10,38 | −25.589 | 10,,37 |
| −13.957 | 3,39 | −13.957 | 3,40 | −20.936 | 7,37 | −20.936 | 7,38 | −13.958 | 24,49 | 13.958 | 23,49 |
| 13.957 | 4,40 | −13.957 | 4,39 | 20.936 | 8,38 | −20.936 | 8,37 | 27.917 | 12,50 | −27.917 | 11,50 |
| −16.283 | 5,37 | −16.283 | 5,38 | 25.589 | 11,37 | −25.589 | 11,38 | 41.873 | 24,51 | −41.873 | 23,51 |
| 16.283 | 6,38 | −16.283 | 6,37 | 25.589 | 12,38 | 25.589 | 12,37 | 2921.283 | 12,0 | −2921.283 | 11,0 |
| 20.936 | 9,37 | −20.936 | 9,38 | −11.631 | 22,49 | 11.631 | 21,49 | −29.609 | 35,0 | −29.609 | 36,0 |
| 20.936 | 10,38 | 20.936 | 10,37 | 23.263 | 10,50 | −23.263 | 9,50 | | | | |
| 23.263 | 11,39 | −23.263 | 11,49 | 34.894 | 22,51 | −34.894 | 21,51 | | | | |
| 23.263 | 12,40 | 23.263 | 12,39 | 2434.403 | 10,0 | −2434.403 | 9,0 | | | | |
| −9.305 | 20,49 | 9.305 | 19,49 | −23.578 | 33,0 | −23.578 | 33,0 | | | | |
| 18.610 | 8,50 | −18.610 | 7,50 | | | | | | | | |
| 27.916 | 20,51 | −27.916 | 19,51 | | | | | | | | |
| 1947.522 | 8,0 | −1947.522 | 7,0 | | | | | | | | |
| −18.643 | 31,0 | −18.643 | 32,0 | | | | | | | | |

TABLE 2 (*continued*).

| i= | 37 | | 38 | | 39 | | 40 | | 41 | | 42 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | C | j,k | C | j,k | C | j,k | C | j,k | C | j,k | C | j,k |
| | 25.589 | 9,35 | 25.589 | 9,36 | 23.262 | 7,35 | 23.262 | 7,36 | 20.936 | 5,35 | 20.936 | 5,36 |
| | 25.589 | 10,36 | −25.589 | 10,35 | 23.262 | 8,36 | −23.262 | 8,35 | 20.936 | 6,36 | −20.936 | 6,35 |
| | 20.936 | 7,33 | 20.936 | 7,34 | 18.610 | 5,33 | 18.610 | 5,34 | 16.283 | 3,33 | 16.283 | 3,34 |
| | 20.936 | 8,34 | −20.936 | 8,33 | 18.610 | 6,34 | −18.610 | 6,33 | 16.283 | 4,34 | −16.283 | 4,33 |
| | 16.283 | 5,31 | 16.283 | 5,32 | 13.957 | 3,31 | 13.957 | 3,32 | 11.632 | 1,31 | 11.632 | 1,32 |
| | 16.283 | 6,32 | −16.283 | 6,31 | 13.957 | 4,32 | −13.957 | 4,31 | 11.632 | 2,32 | −11.632 | 2,31 |
| | 11.631 | 3,29 | 11.631 | 3,30 | 9.305 | 1,29 | 9.305 | 1,30 | −2.326 | 1,27 | −2.326 | 1,28 |
| | 11.631 | 4,30 | −11.631 | 4,29 | 9.305 | 2,30 | −9.305 | 2,29 | 2.326 | 2,28 | −2.326 | 2,27 |
| | 6.979 | 1,27 | 6.979 | 1,28 | −9.305 | 5,25 | 9.305 | 5,26 | 2.326 | 3,25 | 2.326 | 3,26 |
| | 6.979 | 2,28 | −6.979 | 2,27 | −9.305 | 6,26 | −9.305 | 6,25 | −2.326 | 4,26 | 2.326 | 4,25 |
| | −6.979 | 3,25 | 6.979 | 3,26 | −13.958 | 7,27 | 13.958 | 7,28 | −11.632 | 7,25 | 11.632 | 7,26 |
| | −6.979 | 4,26 | −6.979 | 4,25 | −13.958 | 8,28 | −13.958 | 8,27 | −11.632 | 8,26 | −11.632 | 8,25 |
| | −11.631 | 5,27 | 11.631 | 5,28 | −18.610 | 9,29 | 18.610 | 9,30 | −16.284 | 9,27 | 16.284 | 9,28 |
| | −11.631 | 6,28 | −11.631 | 6,27 | −18.610 | 10,30 | −18.610 | 10,29 | −16.284 | 10,28 | −16.284 | 10,27 |
| | −16.284 | 7,29 | 16.284 | 7,30 | −23.263 | 11,31 | 23.263 | 11,32 | −20.937 | 11,29 | 20.937 | 11,30 |
| | −16.284 | 8,30 | −16.284 | 8,29 | −23.263 | 12,32 | −23.263 | 12,31 | −20.937 | 12,30 | −20.937 | 12,29 |
| | −20.936 | 9,31 | 20.936 | 9,32 | −4.652 | 4,49 | 4.652 | 3,49 | −6.979 | 6,49 | 6.979 | 5,49 |
| | −20.936 | 10,32 | −20.936 | 10,31 | 13.957 | 4,51 | −13.957 | 3,51 | 20.936 | 6,51 | −20.936 | 5,51 |
| | −25.589 | 11,32 | 25.589 | 11,34 | 18.610 | 16,52 | −18.610 | 15,52 | 27.915 | 18,52 | −27.915 | 17,52 |
| | −25.589 | 12,34 | −25.589 | 12,33 | 973.761 | 16,0 | −973.761 | 15,0 | 1460.642 | 18,0 | −1460.642 | 17,0 |
| | 2.326 | 2,49 | 2.326 | 1,49 | −41.672 | 39,0 | −41.672 | 40,0 | −44.414 | 41,0 | −44.414 | 42,0 |
| | 6.979 | 2,51 | −6.979 | 1,51 | | | | | | | | |
| | 9.305 | 14,52 | −9.305 | 13,52 | | | | | | | | |
| | 486.881 | 14,0 | −486.881 | 13,0 | | | | | | | | |
| | −40.028 | 37,0 | −40.028 | 38,0 | | | | | | | | |

| i = | 43 | | 44 | | 45 | | 46 | | 47 | | 48 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | C | j,k | C | j,k | C | j,k | C | j,k | C | j,k | C | j,k |
| | 18.610 | 3,35 | 18.610 | 3,36 | 16.284 | 1,35 | 16.284 | 1,36 | −9.305 | 1,33 | −9.305 | 1,34 |
| | 18.610 | 4,36 | −18.610 | 4,35 | 16.284 | 2,36 | −16.284 | 2,35 | 9.305 | 2,34 | −9.305 | 2,33 |
| | 13.958 | 1,33 | 13.958 | 1,34 | −6.979 | 1,31 | −6.979 | 1,32 | −4.652 | 3,31 | −4.652 | 3,32 |
| | 13.958 | 2,34 | −13.958 | 2,33 | 6.979 | 2,32 | −6.979 | 2,31 | 4.652 | 4,32 | −4.652 | 4,31 |
| | −4.652 | 1,29 | −4.652 | 1,30 | −2.326 | 3,29 | −2.326 | 3,30 | 4.652 | 7,27 | 4.652 | 7,28 |
| | 4.652 | 2,30 | −4.652 | 2,29 | 2.326 | 4,30 | −2.326 | 4,29 | −4.652 | 8,28 | 4.652 | 8,27 |
| | 4.652 | 5,25 | 4.652 | 5,26 | 2.326 | 5,27 | 2.326 | 5,28 | 9.305 | 9,25 | 9.305 | 9,26 |
| | −4.652 | 6,26 | 4.652 | 6,25 | −2.326 | 6,28 | 2.326 | 6,27 | −9.305 | 10,26 | 9.305 | 10,25 |
| | −13.958 | 9,25 | 13.958 | 9,26 | 6.979 | 7,25 | 6.979 | 7,26 | −13.873 | 12,49 | 13.873 | 11,49 |
| | −13.958 | 10,26 | −13.958 | 10,25 | −6.979 | 8,26 | 6.979 | 8,25 | 41.873 | 12,51 | −41.873 | 11,51 |
| | −18.610 | 11,27 | 18.610 | 11,28 | −16.284 | 11,25 | 16.284 | 11,26 | 55.831 | 24,52 | −55.831 | 23,52 |
| | −18.610 | 12,28 | −18.610 | 12,27 | −16.284 | 12,26 | −16.284 | 12,25 | 2921.283 | 24,0 | −2921.283 | 23,0 |
| | −9.305 | 8,49 | 9.305 | 7,49 | −11.631 | 10,49 | 11.631 | 9,49 | −59.218 | 47,0 | −59.218 | 48,0 |
| | 27.916 | 8,51 | −27.916 | 7,51 | 34.894 | 10,51 | −34.894 | 9,51 | | | | |
| | 37.221 | 20,52 | −37.221 | 19,52 | 46.526 | 22,52 | −46.526 | 21,52 | | | | |
| | 1947.522 | 20,0 | −1947.522 | 19,0 | 2434.403 | 22,0 | −2434.403 | 21,0 | | | | |
| | −48.252 | 43,0 | −48.252 | 44,0 | −53.187 | 45,0 | −53.187 | 46,0 | | | | |

| i= | 49 | | 50 | | 51 | | 52 | |
|---|---|---|---|---|---|---|---|---|
| | C | j,k | C | j,k | C | j,k | C | j,k |
| | −27.916 | 23,36 | 55.832 | 11,36 | 83.746 | 11,48 | 111.664 | 23,48 |
| | 27.916 | 24,35 | −55.832 | 12,35 | −83.746 | 12,47 | −111.664 | 24,47 |
| | −23.262 | 21,34 | 46.526 | 9,34 | 69.786 | 9,46 | 93.052 | 21,46 |
| | 23.262 | 22,33 | −46.526 | 10,33 | −69.786 | 10,45 | −93.052 | 22,45 |
| | −18.610 | 19,32 | 37.220 | 7,32 | 55.832 | 7,44 | 74.440 | 19,44 |
| | 18.610 | 20,31 | −37.220 | 8,31 | −55.832 | 8,43 | −74.440 | 20,43 |
| | −13.956 | 17,30 | 27.916 | 5,30 | 41.870 | 5,42 | 55.830 | 17,42 |
| | 13.956 | 18,29 | −27.916 | 6,29 | −41.870 | 6,41 | −55.830 | 18,41 |
| | −9.305 | 15,28 | 18.610 | 3,28 | 27.912 | 3,40 | 37.220 | 15,40 |
| | 9.305 | 16,27 | −18.610 | 4,27 | −27.912 | 4,39 | −37.220 | 16,39 |
| | −4.654 | 13,26 | 9.305 | 1,26 | 13.960 | 1,38 | 18.612 | 13,38 |
| | 4.654 | 14,25 | −9.305 | 2,25 | −13.960 | 2,37 | −18.612 | 14,37 |
| | −27.916 | 11,48 | −39.479 | 50,0 | 83.746 | 23,36 | −157.920 | 52,0 |
| | 27.916 | 12,47 | | | −83.746 | 24,35 | | |
| | −23.262 | 9,46 | | | 69.786 | 21,34 | | |
| | 23.262 | 10,45 | | | −69.786 | 22,33 | | |
| | −18.610 | 7,44 | | | 55.832 | 19,32 | | |
| | 18.610 | 8,43 | | | −55.832 | 20,31 | | |
| | −13.956 | 5,42 | | | 41.870 | 17,30 | | |
| | 13.956 | 6,41 | | | −41.870 | 18,29 | | |
| | −9.305 | 3,40 | | | 27.912 | 15,28 | | |
| | 9.305 | 4,39 | | | −27.912 | 16,27 | | |
| | −4.654 | 1,38 | | | 13.960 | 13,26 | | |
| | 4.654 | 2,37 | | | −13.960 | 14,25 | | |
| | −9.870 | 49,0 | | | −88.830 | 51,0 | | |

$$\dot{E}=16.284CD-16.284AF-18.610BG-1947.508\lambda B-18.643E$$

$$\dot{F}=16.284AE+16.284BD-486.877\lambda C-40.028F$$

$$\dot{G}=27.916AD+37.220BE-39.479G,$$

where

$$\lambda=\frac{R}{R_c}.$$

A system of this kind represents the simplest convection model capable of representing non-linear interactions between harmonic components of the stream field. We note that $A$ and $D$ represent the cellular streamline and thermal fields for the Rayleigh critical mode, and $G$ represents the departure of the vertical temperature stratification from the initial linear variation. The allowance of only this single harmonic describing the vertical stratification can be expected to lead to a spurious stable stratification in the center of the fluid (cf., Kuo, 1961).

By way of physical interpretation, the numerical integrations for $\lambda>1$ are imagined to represent an experimental set-up in which the basic vertical temperature contrast is maintained until the perturbation is introduced. In principle, if perturbations are present at all times, the Rayleigh mode would always manifest itself at $\lambda=1$.

Except for very large Rayleigh numbers (e.g., $\lambda>20$) the motions which develop approach a steady cellular form. In the cases of large Rayleigh number, oscillatory, overstable cellular motions are present and, consequently an alternating value of the heat transport about a time-mean value is found. The lack of sufficient degrees of freedom in the vertical undoubtedly contributes to this effect. We shall now be most concerned with the "lower" Rayleigh-number cases, $\lambda \lesssim 10$, which undoubtedly are less seriously affected by the severe truncation embodied by this model.

Fig. 1 is a plot of the steady-state value of $-\Theta(0,2) \equiv -G$ versus $\lambda$. The points represent values obtained from actual integrations. Two regimes are present: for $1\leq\lambda\lesssim 2.125$ the Rayleigh mode (3,1) is present, and for $2.125\lesssim\lambda<10$ a smaller horizontal scale of cellular convection represented by the (4,1) mode is present. It is likely that if the fluid had more degrees of freedom (i.e., components of horizontal wave number greater than 4) it would select these, or combinations of these representing turbulence, rather than continue to select (4,1). The quasi-linear relation between $\lambda$ and $[\Theta(0,2)]_s$ is in good agreement with observations (Malkus, 1954a) and also with the theoretical steady-state results of Malkus and Veronis (1958) and Kuo (1961). As remarked above we should expect to obtain more accurate results if we permit a larger number of degrees of freedom.

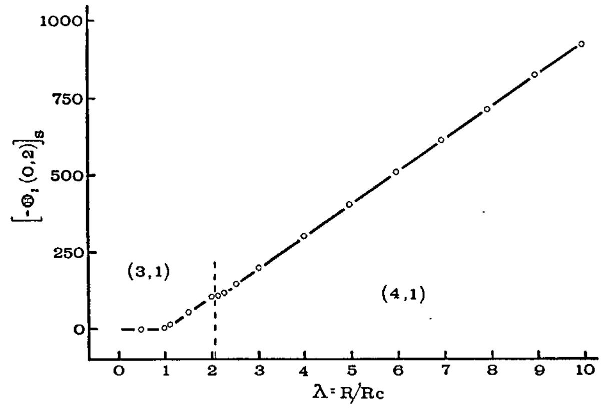


FIG. 1. Steady-state values of $-\Theta(0,2)$ as a function of $\lambda$.

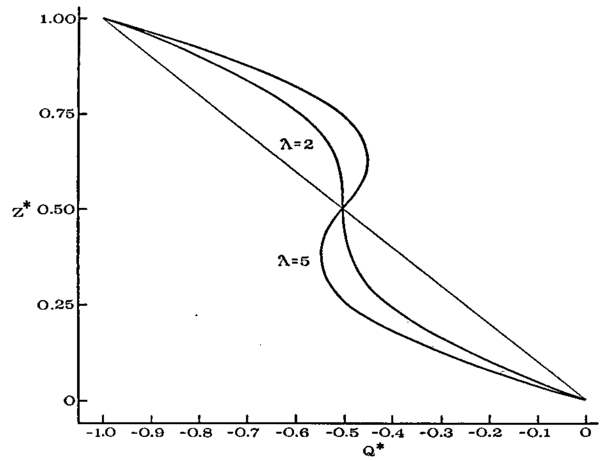


FIG. 2. $Q$ as a function of $z^*$, for $\lambda=2$ and 5.

In Fig. 2 we present a plot of $Q(z^*)$ for $\lambda=2$ and 5. As we remarked above, the lack of resolution in the vertical, due to the inclusion only of a single harmonic to represent the vertical stratification, leads to a reversal in the mean vertical temperature gradient in the center of the fluid. As shown by Kuo (1961) the profile actually takes on an isothermal character there if the higher modes are included, in agreement with observations.

In order to illustrate the transient growth of the perturbations to the steady-states shown in Fig. 1, we have plotted the evolution of $A$, $B$, $C$ and $G$ for a value of $\lambda$ representative of each regime, i.e., for $\lambda=2$ and 5. These are shown in Fig. 3. In both cases we see that one of the components has a maximum growth rate and ultimately establishes itself as the only mode present. We should expect that, if more degrees of freedom (i.e., components) were permitted, at some high value of $\lambda$ several modes would come to coexist in a steady condition at roughly the same amplitudes and this condition would represent thermal turbulence.

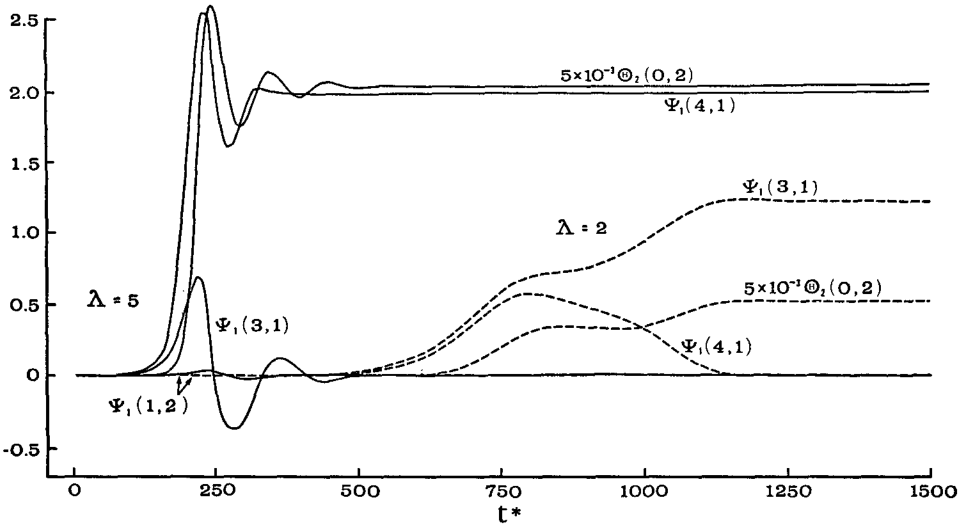


FIG. 3. Evolution of Fourier components. $\lambda=2$ (dashed curves), $\lambda=5$ (solid curves). Initial values in both cases: $\Psi_1(3,1)=\Psi_1(4,1)=\Psi_1(1,2)=0.0005$, $\Theta_2(3,1)=\Theta_2(4,1)=\Theta_2(1,2)=\Theta_2(0,2)=0$.

## 8. Concluding remarks

As noted in the introduction, we view these cases merely as examples to illustrate the methodology. However, in spite of its simplicity the system treated does, in fact, appear to contain a good deal of the real physical content of the problem, especially for low $\lambda$. In order to study real thermal turbulence, however, we must proceed to the consideration of systems of greater complexity. These complexities can be in the form of (1) increased degrees of freedom through inclusion of a greater number of Fourier components. (2) the extension to three dimensions, (3) more realistic boundary conditions, and (4) an expansion of the number of physical ingredients included in the internal dynamics. We hope to make progress in some of these directions in the near future.

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