but we will see in Chapter 6 that when the critical point maps in three iterates to a fixed point for a one-dimensional map, a continuous natural measure is created, which turns out to be desirable for this application.

The three attractors for this system are contained in the union of the three lines. The first, A_1, is the union of two subintervals of the line N_1, and the second, A_2, is the union of two slanted intervals that intersect A_1. The third, A_3, is an "X" at the intersection of the lines N_2 and N_3.

Figure 4.12 shows the basin of infinity in white, and the basins of A_1, A_2, and A_3 in dark gray, light gray, and black. The basins of all three attractors have nonzero area, and are riddled. This means that any disk of nonzero radius in the shaded region, no matter how small, has points from all 3 basins. Proving this fact is beyond the scope of this book. Color Plate 2 is a color version of this figure, which shows more of the detail.

The message of this example is that prediction can be difficult. If we want to start with an initial condition and predict the asymptotic behavior of the orbit, there is no limit to the accuracy with which we need to know the initial condition. This problem is addressed in Challenge 4 in a simpler context: When a basin boundary is fractal, the behavior of orbits near the boundary is hard to predict. A riddled basin is the extreme case when essentially the entire basin is made up of boundary.

4.5 FRACTAL DIMENSION

Our operational definition of fractal was that it has a level of complication that does not simplify upon magnification. We explore this idea by imagining the fractal lying on a grid of equal spacing, and checking the number of grid boxes necessary for covering it. Then we see how this number varies as the grid size is made smaller.

Consider a grid of step-size $1/n$ on the unit interval $[0, 1]$. That is, there are grid points at $0, 1/n, 2/n, \ldots, (n-1)/n, 1$. How does the number of grid boxes (one-dimensional boxes, or subintervals) depend on the step-size of the grid? The answer, of course, is that there are n boxes of grid size $1/n$. The situation changes slightly if we consider the interval $[0, 8]$. Then we need $8n$ boxes of size $1/n$. The common property for one-dimensional intervals is that the number of boxes of size ϵ required to cover an interval is no more than $C(1/\epsilon)$, where C is a constant depending on the length of the interval. This proportionality is often expressed by saying that the number of boxes of size ϵ *scales* as $1/\epsilon$, meaning that the number of boxes is between C_1/ϵ and C_2/ϵ, where C_1 and C_2 are fixed constants not depending on ϵ.

The square $\{(x, y) : 0 \leq x, y \leq 1\}$ of side-length one in the plane can be covered by n^2 boxes of side-length $1/n$. It is the exponent 2 that differentiates this two-dimensional example from the previous one. Any two-dimensional rectangle in $\mathbb{R}^2$ can be covered by $C(1/\epsilon)^2$ boxes of size ϵ. Similarly, a d-dimensional region requires $C(1/\epsilon)^d$ boxes of size ϵ.

The constant C depends on the rectangle. If we consider a square of side-length 2 in the plane, and cover by boxes of side-length $\epsilon = 1/n$, then $4(1/\epsilon)^2$ boxes are required, so $C = 4$. The constant C can be chosen as large as needed, as long as the scaling $C(1/\epsilon)^2$ holds as ϵ goes to 0.

We are asking the following question. Given an object in m-dimensional space, how many m-dimensional boxes of side-length ϵ does it take to cover the object? For example, we cover objects in the plane with $\epsilon \times \epsilon$ squares. For objects in three-dimensional space, we cover with cubes of side ϵ. The number of boxes, in cases we have looked at, comes out to $C(1/\epsilon)^d$, where d is the number we would assign to be the dimension of the object. Our goal is to extend this idea to more complicated sets, like fractals, and use this "scaling relation" to *define* the dimension d of the object in cases where we don't start out knowing the answer.

Notice that an interval of length one, when viewed as a subset of the plane, requires $1/\epsilon$ two-dimensional boxes of size ϵ to be covered. This is the same scaling that we found for the unit interval considered as a subset of the line, and matches what we would find for a unit interval inside $\mathbb{R}^m$ for any integer m. This scaling is therefore intrinsic to the unit interval, and independent of the space in which it lies. We will denote by $N(\epsilon)$ the number of boxes of side-length ϵ needed to cover a given set. In general, if S is a set in $\mathbb{R}^m$, we would like to say that S is a d-dimensional set when it can be covered by

$$N(\epsilon) = C(1/\epsilon)^d$$

boxes of side-length ϵ, for small ϵ. Stated in this way, it is not required that the exponent d be an integer.

Let S be a bounded set in $\mathbb{R}^m$. To measure the dimension of S, we lay a grid of m-dimensional boxes of side-length ϵ over S. (See Figure 4.13.) Set $N(\epsilon)$ equal to the number of boxes of the grid that intersect S. Solving the scaling law for the

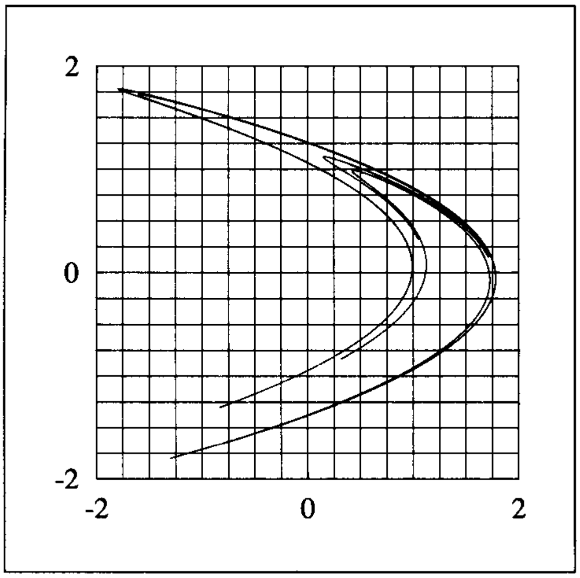

Figure 4.13 Grid of boxes for dimension measurement.
The Hénon attractor of Example 4.10 is shown beneath a grid of boxes with side-length $\epsilon = 1/4$. Of the 256 boxes shown, 76 contain a piece of the attractor.

dimension d gives us

$$d = \frac{\ln N(\epsilon) - \ln C}{\ln(1/\epsilon)}.$$

If C is constant for all small ϵ, the contribution of the second term in the numerator of this formula will be negligible for small ϵ. This justifies the following:

Definition 4.14 A bounded set S in $\mathbb{R}^n$ has **box-counting dimension**

$$\text{boxdim}(S) = \lim_{\epsilon \to 0} \frac{\ln N(\epsilon)}{\ln(1/\epsilon)},$$

when the limit exists.

We can check that this definition of dimension gives the correct answer for a line segment in the plane. Let S be a line segment of length L. The number of boxes intersected by S will depend on how it is situated in the plane, but roughly speaking, will be at least L/ϵ (if it lies along the vertical or the horizontal) and no more than $2L/\epsilon$ (if it lies diagonally with respect to the grid, and straddles pairs of neighboring boxes). As we expect, $N(\epsilon)$ scales as $1/\epsilon$ for this one-dimensional set. In fact, $N(\epsilon)$ is between L times $1/\epsilon$ and $2L$ times $1/\epsilon$. This remains true for infinitesimally small ϵ. Then Definition 4.14 gives $d = 1$.